Math 173 Exam 3 Name Roy

Show all work for full credit in the space provided. Sketches are encouraged but not required and may be used for partial credit.

Problems 1 – 5: 16 points each. Problems 6 – 7: 10 points each.

1. Use LaGrange multipliers to find the maxima and minima of $f(x, y) = x^2 + 2y^2 - 4y$ subject to the constraint $x^2 + y^2 = 9$ (3)

$\nabla f = \langle 2x, 4y-4 \rangle \quad \nabla g = \langle 2x, 2y \rangle$

$\nabla f = \lambda \nabla g \Rightarrow$ (1) $2x = 2\lambda x \Rightarrow \cancel{2\lambda x - 2x = 0 \Rightarrow 2x(\lambda - 1)} = 0$

$x = 0$ or $\lambda = 1$

(2) $4y - 4 = 2\lambda y \Rightarrow 2y(2 - \lambda) = 4$

$\rightarrow$ If $x = 0$, in (3) $0^2 + y^2 = 9$, $y = \pm 3$

$f(0,3) = 2 \cdot 3^2 - 4(3) = 18 - 12 = 6$

$f(0,-3) = 2 \cdot 9 - 4(-3) = 18 + 12 = 30$

$\rightarrow$ If $\lambda = 1$, in (2) $2y(2-1) = 4$, $y = 2$

in (3) $x^2 + 2^2 = 9$, $x = \pm\sqrt{5}$

$x^2 = 5$

$f(+\sqrt{5}, 2) = 5 + 2(2)^2 - 4(2) = 5 + 8 - 8 = 5$

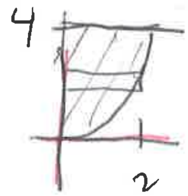
$f(-\sqrt{5}, 2) = 5 + 8 - 8 = 5$

Critical:

(x,y)	$f(x,y)$	
$(0,3)$	6	
$(0,-3)$	30	$\leftarrow$ max
$(\sqrt{5},2)$	5	$\leftarrow$ min
$(-\sqrt{5},2)$	5	

This is too difficult. If you switch order! as is.

2. Evaluate the integral $\int_0^2 \int_{x^2}^4 x\boxed{e^{y^2}\,dy}\,dx$



$y: [x^2, 4] \Rightarrow y = x^2 \Rightarrow x = \sqrt{y}$ (+ only)

$x\,[0,2]$ $\quad y = 4$

$$\int_0^4 \int_0^{\sqrt{y}} x e^{y^2}\,dx\,dy$$

$e^u \quad \frac{1}{2}du$

Points deducted

No drawing	Drawing
−0 |	−0 ~~||||~~ ~~||||~~ |
	−0 ||
−6	−3
−8 ||	−8 |
−10 ~~||||~~ ~~||||~~ |	−10 |
−12 ||| ||	

$$= \int_0^4 \frac{x^2}{2} e^{y^2}\Big]_0^{\sqrt{y}} dy = \int_0^4 \frac{y}{2} e^{y^2}\,dy$$

let $u = y^2$

$du = 2y\,dy$

$y\,dy = \frac{1}{2}du$

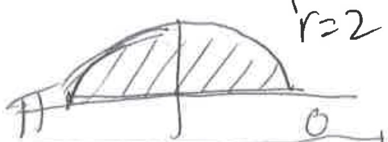
$$= \frac{1}{2}\cdot\frac{1}{2}\left[e^{y^2}\right]_0^4$$

$$= \frac{1}{4}(e^{16} - e^0) = \frac{1}{2}(e^{16} - 1)$$

$r\sin\theta \quad r^2$

3. Evaluate the integral $\iint_D y\cdot(x^2+y^2)\,dA$, where D is the half circle enclosed by $y = \sqrt{4 - x^2}$ and the line $y = 0$.

$r = 2$



$$\int_0^{\pi}\int_0^2 r\sin\theta \cdot r^2 \cdot r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^2 r^4 \sin\theta\,dr\,d\theta$$

$$= \int_0^{\pi} \sin\theta\,d\theta \int_0^2 r^4\,dr$$

$$= -\cos\theta\Big]_0^{\pi} \cdot r^5/5\Big]_0^2$$

$$= -(-1-1)\cdot(32/5 - 0)$$

$$= 2\cdot 32/5 = 64/5$$

In rectangular - not recommended

$$\int_{-2}^2 \int_0^{\sqrt{4-x^2}} yx^2 + y^3\,dy\,dx$$

$$= \int_{-2}^2 \frac{[4-x^2]\cdot x^2}{2} + \frac{(4-x^2)^2}{4}\,dx$$

$$= \int_{-2}^2 2x^2 - \frac{x^4}{2} + 4 - \cancel{2x^2} + \frac{x^4}{4}\,dx$$

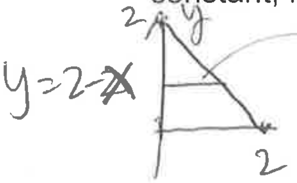
$$= -\frac{x^5}{10} + \frac{x^3}{3} + 4x\Big]_{-2}^2$$

$$= -\frac{32}{10} + \frac{8}{3} + 8 + 8 + \frac{8}{3} - \frac{32}{10}$$

$$= 16 + \frac{16}{3} - \frac{64}{10} = \frac{384}{30}$$

$$= \frac{64}{5}$$ ugh!

4. a) Find the mass of a triangular lamina with vertices at (0,0), (0,2) and (2,0) if the density function is proportional to the distance from the y-axis. Include the proportionality constant, k.



distance = x

$\rho(x,y) = kx$

$$\text{mass} = \int\int \rho(x,y)\,dA$$

$$= \int_0^2 \int_0^{2-x} kx \, dy\, dx$$

$$= \int_0^2 kxy \Big]_{y=0}^{y=2-x} dx$$

$$= k\int_0^2 x(2-x)\,dx$$

$$= k\left[x^2 - x^3/3\right]_0^2$$

$$= k[4 - 8/3] = \frac{4}{3}k$$

b) Using the mass from part a), set up the integral for $\bar{x}$, the x-coordinate of the center of mass. Include limits of integration, and simplify constants, but DO NOT INTEGRATE (no bonus for this problem).

$$\bar{X} = \frac{1}{m} \int_0^2 \int_0^{2-x} x \cdot \underbrace{kx}_{\rho(x,y)} \, dy\, dx$$

$$= \frac{3}{4\cancel{k}} \int_0^2 \int_0^{2-x} \cancel{k} x^2 \, dy\, dx$$

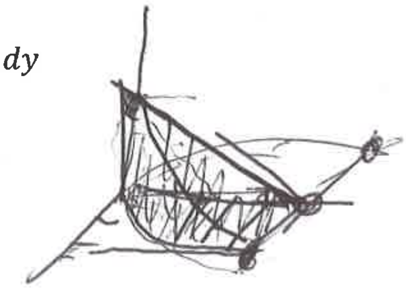
$$= \frac{3}{4} \int_0^2 \int_0^{2-x} x^2 \, dy\, dx$$

The problem was supposed to look like this! $2-\frac{1}{2}y$

$z = 2 - \frac{1}{2}y$

5. For the integral $\int_0^2 \int_{x^2}^4 \int_0^{2-y} f(x,y,z)\,dz\,dy\,dx$ find the correct limits for the new orders of integration below, where V is the volume defined by the original integral. DO NOT INTEGRATE.

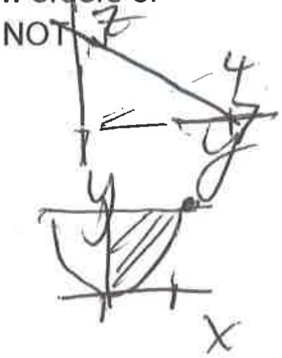
a) $\iiint_V f(x,y,z)\,dz\,dx\,dy$



$z: [0, 2-\frac{y}{2}]$

$y: [x^2, 4]$

$x: [0, 2]$



$y = x^2;\ x = \sqrt{y}$

$$\int_0^4 \int_0^{\sqrt{y}} \int_0^{2-\frac{y}{2}} f(x,y,z)\,dz\,dx\,dy$$

$z = 2 - \frac{1}{2}y,$

$y = 4 - 2z$

b) $\iiint_V f(x,y,z)\,dx\,dy\,dz$

$x = \sqrt{y};$

$$\int_0^2 \int_0^{4-2z} \int_0^{\sqrt{y}} \cancel{dz}\,dy\,dz$$

$z = 2 - y$

$y = 2 - z$

$z = 2 - \frac{1}{2}y$

$$\int_0^2 \int_0^{2-\cancel{x}^2/2} \int_{x^2}^{4} dy\,dz\,dx$$

$y[\sqrt{x}, 4]$

The bonus part is a lovely work of mathematical art.

6. Set up the integral needed to find the surface area of the part of the half-cylinder $x^2 + z^2 = 1$ that lies above the rectangle $0 \le x \le \frac{1}{2}$ and $0 \le y \le 1$ in the x-y plane. DO NOT INTEGRATE (except for bonus).

$$z = +\sqrt{1-x^2}$$

$$\partial z/\partial x = -2x \cdot \tfrac{1}{2}(1-x^2)^{-1/2} = \frac{-x}{\sqrt{1-x^2}}$$

$$\partial z/\partial y = 0$$

$$SA = \int_{y=0}^{y=1} \int_{x=0}^{x=1/2} \sqrt{1+\frac{x^2}{1-x^2}}\, dx\, dy$$

← This is ok

$$= \int_0^1 \int_0^{1/2} \sqrt{\frac{1-\cancel{x^2}+\cancel{x^2}}{1-x^2}}\, dx\, dy = \int_0^1 \int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\, dx\, dy$$

Bonus portion (4 pts) Evaluate the integral above. You must start with the correct integral (above) to earn bonus points.

memorize $d \sin^{-1}x/dx$

or trig subst

$x = \sin\theta$

$dx = \cos\theta\, d\theta$

$\theta = \sin^{-1} x$

$$\int_0^1 1\, dy \int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\, dx$$

$$= y\Big]_0^1 \cdot \int_{\sim}^{\sim} \frac{\cos\theta\, d\theta}{\sqrt{\cos^2\theta}\, d\theta} \quad = 1$$

$$= 1 \cdot \theta\Big]_{\sim}^{\sim} = 1 \cdot \sin^{-1} x\Big]_0^{1/2}$$

$$= \sin^{-1}\tfrac{1}{2} - \sin^{-1} 0$$

$$= \pi/6$$

The bonus is an ugly jumble of a construction site,

7. Set up the integral with appropriate limits. but DO NOT INTEGRATE (except for bonus): $\iiint_E (x-y)\,dV$, where E is the solid that lies between the cylinders $x^2+y^2=1$ and $x^2+y^2=16$ above the x-y plane, and below the plane $z=y+4$

$r: [1,4]$

$\theta: [0, 2\pi]$

$z: [0, y+4] \to [0, r\sin\theta + 4]$

$$\int_0^{2\pi}\int_1^4\int_{z=0}^{z=r\sin\theta+4} (r\cos\theta - r\sin\theta)\, r \, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_1^4\int_0^{r\sin\theta+4} r^2(\cos\theta - \sin\theta)\, dz\, dr\, d\theta$$

Bonus portion (4 pts) Evaluate the integral above. You must start with the correct integral above to earn bonus points.

$$\int_0^{2\pi}\int_1^4 z\cdot r^2(\cos\theta - \sin\theta)\, dr\, d\theta$$

product is 4 terms!

$$\int_0^{2\pi}\int_1^4 r^2(r\sin\theta + 4)(\cos\theta - \sin\theta)\, dr\, d\theta$$

$$\int_0^{2\pi}\int_1^4 r^2[r\sin\theta\cos\theta - r\sin^2\theta + 4\cos\theta - 4\sin\theta]\, dr\, d\theta$$

$$\int_0^{2\pi} \left[\frac{r^4}{4}\underbrace{\sin\theta\cos\theta}_{\to\frac{1}{2}\sin 2\theta} - \frac{r^4}{4}\underbrace{\sin^2\theta}_{\frac{1}{2}(1-\cos 2\theta)} + \frac{4r^3}{3}\cos\theta - \frac{4r^3}{3}\sin\theta\right]_1^4 d\theta$$

or $\sin^2\theta$

$$\int_0^{2\pi}\left[\frac{255}{4}\sin\theta\cos\theta - \frac{255}{4}\sin^2\theta + \frac{252}{3}\cos\theta - \frac{252}{3}\sin\theta\right] d\theta$$

$$\left.\frac{255}{4}\cdot\frac{1}{2}\left(\frac{-\cos 2\theta}{2}\right) - \frac{255}{4}\cdot\frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + 84\sin\theta + 84\cos\theta\right]_0^{2\pi}$$

$$\cancel{\frac{255}{16}(-1)(-1+1)} - \frac{255}{8}[(2\pi)+0] + \frac{252}{3}[0] + \cancel{\frac{252}{3}[1-1]} \to 0$$

$$= -\frac{255}{8}\cdot 2\pi = -\frac{255\pi}{4}$$