Yellow version and tan version

Math 173 – Exam 2 Name

Each problem is worth 16 points, except # 7, which is worth 4 points.

1. For parts a) and b), determine whether the following limit exists, and prove your conclusion.

a) $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^4+y^2}$

let $y = x^2$:

$\lim_{(x,x^2)\to(0,0)} \frac{3x^2 \cdot x^2}{x^4+x^4} = \lim_{x\to 0} \frac{3\cancel{x^4}}{2\cancel{x^4}} = \frac{3}{2}$

DNE

let $x=0$

$\lim_{(0,y)\to(0,0)} \frac{3\cdot 0y}{0+y^2} = 0$

b) $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$

≤ 1

$0 \le \lim_{(x,y)\to(0,0)} |3y| \left|\frac{x^2}{x^2+y^2}\right| \le \lim_{(x,y)\to 0,0} |3y| = 0$

Since $0 \le \lim_{(x,y)\to(0,0)} |f(x)| \le 0$, $\lim_{(x,y)\to(0,0)} f(x) = 0$

(By squeeze Theorem)

c) Determine whether the function below is continuous at the origin. Be sure to give your reasoning.

$$f(x,y) = \begin{cases} \dfrac{3x^2y}{x^2+y^2} & for\ (x,y) \neq (0,0) \\ 0 & for\ (x,y) = (0,0) \end{cases}$$

① $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+1} = 0$ (call it L) from b) $\Rightarrow$ the limit exists at $(0,0)$

② $f(0,0) = 0 \Rightarrow f(0,0)$ exists

③ $L = f(0,0) = 0$ $\therefore f(x,y)$ is continuous at $(0,0)$

2. For $f(x,y) = 3x^2 + 2x - y^2$

a) Find $\nabla f(x,y)$

$$\nabla f = \langle f_x, f_y \rangle$$
$$= \langle 6x+2, -2y \rangle$$

$$\nabla f(2,1) = \langle 6(2)+2, -2(1) \rangle$$
$$= \langle 14, -2 \rangle$$

b) Find the directional derivative of f at (2, 1) in the direction of $\vec{v} = -\hat{\imath} + 3\,\hat{\jmath}$

$$\vec{V} = \langle -1, 3 \rangle \qquad \vec{u} = \frac{\vec{V}}{|\vec{V}|} = \frac{\langle -1, 3 \rangle}{\sqrt{1+9}} = \left\langle \frac{-1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle$$

$$D_{\vec{u}} f(2,1) = \nabla f \cdot u = \langle 14, -2 \rangle \left\langle -\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle$$
$$= -\frac{14}{\sqrt{10}} - \frac{6}{\sqrt{10}} = -\frac{20}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}}$$
$$= -2\sqrt{10}$$

c) Find the equation of the tangent plane to the function at the point (2, 1)

∇f is vector normal to tangent plane

$$z - z_0 = f_x(x - x_0) + f_y(y - y_0)$$
$$z_0 = f(2,1) = 3(2)^2 + 2(2) - 1^2$$
$$= 12 + 4 - 1 = 15$$
$$z - 15 = 14(x - 2) - 2(y - 1)$$
$$14x - 28 - 2y + 2 - z + 15 = 0$$
$$14x - 2y - z = 11$$

3. For the function $f(x,y) = 4 + x^3 + y^3 - 3xy$:
   a) Find all critical values and classify (local max, local min, saddle point).

$f_x = 3x^2 - 3y = 0 \Rightarrow y = x^2$

$f_y = 3y^2 - 3x = 0 \Rightarrow x = y^2$

2 imaginary roots

$y=0 \quad y=1$

Subst: $y = (y^2)^2 \Rightarrow y^4 - y = y(y^3 - 1) = y(y-1)(y^2+y+1)$

	crit pts	D	
$y=0,\ x=0^2=0$	$(0,0)$	$\leftarrow 0-9=-9 \leftarrow$	Saddle pt
$y=1,\ x=1^2=1$	$(1,1)$	$\leftarrow 36-9=27$	Local min

$f_{xx} = 6 > 0$

(face up

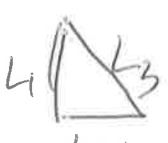
classify:

$f_{xx} = 6x \qquad D = \begin{vmatrix} 6x & -3 \\ -3 & 6y \end{vmatrix}$

$f_{yy} = 6y$

$f_{xy} = -3 \qquad = 36xy - 9$

b) Find the absolute maximum and minimum of the function defined on the triangle bounded by the lines x = 0, y = 0, and y = 3 – x



$L_1: x=0 \quad f_y(0,y) = 3y^2 = 0,\ y=0,\ x=0$ (same as above)

$L_2: y=0 \quad f_x(0,y) = 3x^2 = 0;\ x=0,\ y=0$ (same)

$L_3: y = 3-x \qquad (3-x)(3-x)(3-x)$

$(3-x)^3 = 3^3 - 3\cdot 3^2 x + 3\cdot 3^1 x^2 - x^3$

$f(x, 3-x) = 4 + x^3 + (3-x)^3 - 3x(3-x)$

$= 4 + \cancel{x^3} + (27 - 27x + 9x^2 - \cancel{x^3}) - 9x + 3x^2$

$f = 12x^2 - 36x + 31$

$f_x = 24x - 36 = 0$

$x = 36/24 = 3/2$

$y = 3 - 3/2 = 3/2$

$f(3/2)(3/2) = 4 + 27/8 + 27/8 - 27/4 = 4$

$f(3,0) = 4 + 27 = 31$

$f(0,3) = 4 + 27 = 31$

$f(0,0) = 4 + 0 = 4$

$f(1,1) = 4 + 1 + 1 - 3 = 3$

(x,y)	$f(x,y)$
$(0,0)$	4
$(1,1)$	3 Abs Min
$(3/2, 3/2)$	4
$(3,0)$	31 Abs. max
$(0,3)$	31

Vertices

4. Wave heights, $h(v,t)$, are dependent on wind speed (velocity), and the time the wind has been blowing at that speed, as shown in the table:

Wave Height (feet)

Duration (hours)

Wind speed (knots) v \ t	5	10	15	20	30	40	50
10	2	2	2	2	2	2	2
15	4	4	5	5	5	5	5
20	5	7	8	8	9	9	9
30	9	13	16	17	18	19	19
40	14	21	25	28	31	33	33
50	19	29	36	40	45	48	50
60	24	37	47	54	62	67	69

a) Estimate $h_v(40,10)$

① $\frac{\partial h}{\partial v} \approx \frac{\Delta h}{\Delta v} = \frac{21-13}{40-30} = \frac{8}{10} = \frac{4}{5}$ or $.8$

② $\frac{\partial h}{\partial v} \approx \frac{\Delta h}{\Delta v} = \frac{29-21}{50-40} = \frac{8}{10} = \frac{4}{5}$ or $.8$

avg. is $.8$

b) Estimate $h_t(40,10)$

① $\partial h/\partial t \approx \Delta h/\Delta t = \frac{21-14}{10-5} = \frac{7}{5} = 1.4$

② $\partial h/\partial t \approx \Delta h/\Delta t = \frac{25-21}{15-10} = \frac{4}{5} = 0.8$

avg. is 1.1

c) Use linearization to estimate the wave height at v= 38, t = 11.

$h(38,11) \approx h(40,10) + .8(38-40) + 1.1(11-10)$

$= 21 - 1.6 + 1.1 = 20.5$

5. Consider the curve defined by $\boldsymbol{r}(t) = \langle \sqrt{2} \cdot t, e^t, e^{-t} \rangle$

ln 3 for tan. version

a) Find the arc length of $\boldsymbol{r}(t)$ for $0 \le t \le \ln 7$ Hint: $a^2 + 2ab + b^2 = ?$

$$S = \int_0^{\ln 7} \sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2}$$

$$= \int_0^{\ln 7} \sqrt{(\sqrt{2})^2 + (e^t)^2 + (e^{-t})^2}\, dt$$

$$= \int_0^{\ln 7} \sqrt{2 + e^{2t} + e^{-2t}}\, dt = \int_0^{\ln 7} \sqrt{(e^t + e^{-t})^2}\, dt$$

$$= \int_0^{\ln 7} e^t + e^{-t}\, dt = e^t - e^{-t} \Big]_0^{\ln 7} = e^{\ln 7} - e^{-\ln 7} - (1-1)$$

$$= 7 - 1/7 = 6\tfrac{6}{7} = \tfrac{48}{7}$$

Tan version: $3 - \tfrac{1}{3} = \tfrac{8}{3} = 2\tfrac{2}{3}$

b) Find the curvature of $\boldsymbol{r}(t)$ at t = 0

$$r' = \langle \sqrt{2}, e^t, -e^{-t} \rangle = \langle \sqrt{2}, 1, -1 \rangle$$

$$r'' = \langle 0, e^t, e^{-t} \rangle = \langle 0, 1, 1 \rangle$$

$$r' \times r'' = \begin{vmatrix} i & j & k \\ \sqrt{2} & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = (1+1)i - \sqrt{2}j + \sqrt{2}k$$

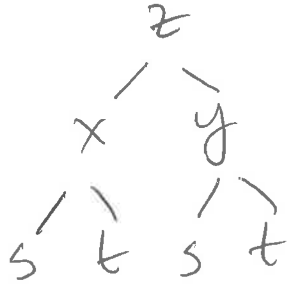
$$= 2i - \sqrt{2}j + \sqrt{2}k$$

$$K = \frac{|r' \times r''|}{|r'|^3} = \frac{\sqrt{4+2+2}}{(\sqrt{2+1+1})^3} = \frac{\sqrt{8}}{2^3} = \frac{2\sqrt{2}}{8}$$

$$= \frac{\sqrt{2}}{4}$$

Eval: $x = 1^2 + 0^2 = 1$ $\quad y = 1 \cdot 0 = 0$

6. a) For $z = e^x \tan y$, where $x = s^2 + t^2$ and $y = st$, find $\partial z/\partial t$ when $s = 1, t = 0$



eval

$z_x = e^x \tan y = e^1 \cdot 0 = 0$

$z_y = e^x \sec^2 y = e^1 \cdot 1 = e$

$x_t = 2t = 2(0) = 0$

$y_t = s = 1$

$$\frac{\partial z}{\partial t} = z_x \cdot x_t + z_y \cdot y_t = 0 \cdot 0 + e \cdot 1 = e$$

b) For the equation $x^2 + y^2 - 2xy + 6x - 4y - 7 = 0$, calculate dy/dx and evaluate at (0, 1) if y is a function of x in the equation

$$dy/dx = \frac{-F_x}{F_y} = -\frac{(2x - 2y + 6)}{(2y - 2x - 4)}$$

eval at (0,1)

$$= \frac{-(-2+6)}{2-4} = \frac{-4}{-2} = 2$$

7. (4 pts) Describe or sketch the domain of $f(x, y) = \arcsin(x^2 + y^2)$:

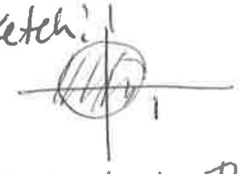
$z = \arcsin(x^2 + y^2)$

$(\sin z = \sin \arcsin(x^2 + y^2) = x^2 + y^2$

$-1 \le \sin z \le 1$

so $x^2 + y^2 \le 1$;

Sketch:



or $\{(x, y) \mid x^2 + y^2 \le 1\}$

Describe: The interior of a circle with radius ≤ 1