Math 173 – Exam 2 Name

# 1-2: 10 points each; # 3-6: 12 points each, #7-8: 16 points each

1. The acceleration of a particle is given by $\boldsymbol{a}(t) = \langle \sin t, e^t, 3t^2 \rangle$ . Find the velocity function, $\boldsymbol{v}(t)$ if the initial velocity is $\boldsymbol{v}(0) = \langle -2, 5, 3 \rangle$

$$\vec{V}(t) = \int \vec{a}(t) + C$$

$$v(t) = \langle -\cos t + C_1,\ e^t + C_2,\ t^3 + C_3 \rangle$$

$$v(0) = \langle -1 + C_1,\ 1 + C_2,\ C_3 \rangle$$

$$= \langle -2,\ 5,\ 3 \rangle$$

$$C_1 = -2+1 = -1 \qquad C_2 = 5-1 = 4 \qquad C_3 = 3$$

Final velocity function:

$$\vec{V}(t) = \langle -\cos t - 1,\ e^t + 4,\ t^3 + 3 \rangle$$

2. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the equation $z = yz + x\,lny$ ; $z(1-y) = x \ln y$

Implicit diff: remember that $z$ is a function dependent on $x$ and $y$ ; $z = \frac{x \ln y}{1-y}$

① Derive wrt $x$ :

$$\partial z/\partial x = y \cdot \partial z/\partial x + \ln y$$

$$\partial z/\partial x (1-y) = \ln y \ ; \ \partial z/\partial x = \frac{\ln y}{1-y}$$

② Derive wrt $y$ — product rule

$$\partial z/\partial y = y \cdot \partial z/\partial y + z + x \cdot \frac{1}{y}$$

$$\partial z/\partial y (1-y) = z + \frac{x}{y} \ ; \ \partial z/\partial y = \frac{zy + x}{y(1-y)}$$

subst. for $z$

+1 bonus if →

$$= \frac{xy \ln y - xy + x}{y(1-y^2)}$$

3. Determine whether the following limits exist, and prove your conclusion.

a) $\lim_{(x,y)\to(1,1)} \frac{x-y}{x^2-y^2}$

(limit exists)

$$\lim_{(x,y)\to(1,1)} \frac{\cancel{x-y}}{(\cancel{x-y})(x+y)} = \lim_{(x,y)\to(1,1)} \frac{1}{x+y} = \frac{1}{2}$$

removable singularity

direct substitution

b) $\lim_{(x,y)\to(0,0)} \frac{5xy}{x^2+y^2}$

Along line $y=x$: $\lim_{(x,x)\to(0,0)} \frac{5x\cdot x}{x^2+x^2} = \frac{5\cancel{x^2}}{2(\cancel{x^2})} = \frac{5}{2}$

Along line $y=0$ $\lim_{(x,0)\to(0,0)} \frac{5x\cdot 0}{x^2+0^2} = \lim_{(x,0)\to 0,0} 0 = 0$

Limit DNE (different values from different approaches

4. a) For $f(x,y) = 3x^2 + 2x - y^2$, find the equation of the tangent plane at the point (1, -2, 1)

By gradient

$z = 3x^2+2x-y^2$

$g(x,y,z) = 3x^2+2x-y^2-z$

$g_x = 6x+2 = 8$ at $1,-2,1$

$g_y = -2y = -2(-2) = 4$

$g_z = -1$

$8(x-1)+4(y+2)-(z-1)=0$

$8x+4y-z=-1$

By $f(x,y)$ at $(1,-2,1)$

$f_x = 6x+2 = 6\cdot 1+2 = 8$

$f_y = -2y = -2(-2) = 4$

$8(x-1)+4(y+2) = z-1$

$8x-8+4y+8 = z-1$

$8x+4y-z = -1$

or $z = 8x+4y+1$

b) Use the linearization of $f$ to approximate $f$ at the point (1.1, -1.9)

$L(x,y) = f(1,-2) + f_x(x-1) + f_y(x+2)$

$= 1+8(1.1-1)+4(-1.9+2) = 1+.8+.4$

$= 2.2$

5. Find and identify (local max, local min, or saddle point) all critical points of $f(x, y) = x^3 - 3x + 3xy^2$

$f_x = 3x^2 - 3 + 3y^2$

$f_y = 6xy = 0 \Rightarrow x = 0$ or $y = 0$

Critical pts

① $x = 0$; $f_x = 0 \to -3 + 3y^2 = 0$; $y^2 = 1$, $y = \pm 1$

(1a) $x = 0, y = 1$ $\quad (0, 1)$

(1b) $x = 0, y = -1$ $\quad (0, -1)$

② $y = 0$, $f_x = 0 \Rightarrow 3x^2 - 3 = 0$; $x^2 = 1$, $x = \pm 1$

(2a) $x = 1, y = 0$ $\quad (1, 0)$

(2b) $x = -1, 1 = 0$ $\quad (-1, 0)$

What kind?

$f_{xx} = 6x \qquad f_{yy} = 6x \qquad f_{xy} = 6y$

$$D = \begin{vmatrix} 6x & 6y \\ 6y & 6x \end{vmatrix} = 36x^2 - 36y^2$$

At $(0,1)$, $D = -36 < 0$ saddle pt

At $(0,-1)$ $D = -36 < 0$ saddle pt

At $(1,0)$ $D = 36 > 0$ max or min

$f_{xx} = 6 > 0 \to$ concave up $\to$ local minimum

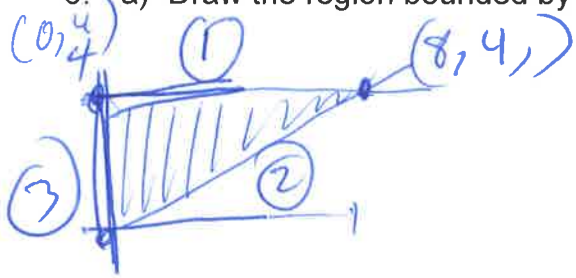
$f(1,0) = 1 - 3 = -2$

At $(-1,0)$ $D = 36(-1)^2 > 0$ max or min

$f_{xx} = -6 < 0$ concave down $\to$ local max

$f(-1,0) = -1 - (-3) = 2$

6. a) Draw the region bounded by the curves $y = 4$, $y = \frac{1}{2}x$, and $x = 0$.




b) Find the absolute maximum and absolute minimum of the function $f(x, y) = 2x - y$ over the region above.

$f_x = 2$, $f_y = -1 \Rightarrow$ no local max or min

| $(x,y)$ | $f(x,y)$ | |
|---|---|---|
| $(0,4)$ | $-4$ | Abs. min |
| $(8,4)$ | $12$ | Abs. max |
| $(0,0)$ | $0$ | |

① $L_1 =$ $y = 4$, $f(x,4) = 2x - 4$

strictly increasing, test end pts

$x = 0$: $f(0,4) = 0 - 4 = -4$

$x = 8$: $f(8,4) = 2(8) - 4 = 12$

② $L_2 =$ $y = \frac{1}{2}x$ $\quad f(x, \frac{1}{2}x) = 2x - \frac{1}{2}x = \frac{3}{2}x$

strictly increasing, test end pts

$(8,4)$ already tested

$(0,0)$: $f(0,0) = 0$

③ $L_3 \Rightarrow x = 0 \Rightarrow$ constant, test end pts.

$(0,0)$ already tested

$(0,4)$ already tested

Abs max is 12 at $(8,4)$

Abs. min is $-4$ at $(0,4)$

7. A rectangular plate in the x-y plane is has temperature readings for $0 \le x \le 8$, and $0 \le y \le 6$, x & y in meters, in gray strips. The table is aligned with the first quadrant.

| | | | | | |
|---|---|---|---|---|---|
| 6 | 51 | 62 | 68 | 75 | 80 |
| 4 | 45 | 60 | 72 | 80 | 86 |
| 2 | 38 | 56 | 74 | 87 | 90 |
| 0 | 30 | 52 | 78 | 98 | 96 |
| y x | 0 | 2 | 4 | 6 | 8 |

a) Estimate the values of the partial derivatives $T_x\,(6,4)$ $and$ $T_y\,(6,4)$

4 ① above $\frac{\Delta T}{\Delta x} = \frac{86-80}{8-6} = \frac{6}{2} = 3$ and ② below $\frac{\Delta T}{\Delta x} = \frac{80-72}{6-4} = \frac{8}{2} = 4$

① above Average is 3.5; $T_x \simeq 3.5$

4 $\frac{\Delta T}{\Delta y} = \frac{75-80}{6-4} = \frac{-5}{2} = -2.5$, ② below $\frac{\Delta T}{\Delta y} = \frac{80-87}{4-2} = \frac{7}{2} = -3.5$

Average is $-3$ $T_y \approx -3$

b) Write the gradient vector, $\nabla T$

2 $\nabla T = \langle 3.5, -3 \rangle$

c) For the unit vector $\boldsymbol{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$ find $D_u T(6,4)$, the directional derivative of $\boldsymbol{u}$ at the point (6,4).

2 $D_u T(6,4) = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle \langle 3.5, -3 \rangle = \frac{3.5}{\sqrt{2}} - \frac{3}{\sqrt{2}} = \frac{1/2}{\sqrt{2}} = \frac{1}{2\sqrt{2}}$

$\approx 0.35$

d) Interpret the meaning of $T_x\,(6,4)$ $and$ $T_y\,(6,4)$ and $D_u T(6,4)$

4 $T_x$ – change in temp. as x increases near (6,4)
– moving "sideways", temp. increase $\approx 3.5°$/meter

$T_y$ – change in temp as y increases near (6,4).
– moving "up", temp. decreases $\approx 3°$/meter

$D_u T(6,4)$ – change in temp as both x and y increase in a "north east" direction
– moving "northeast", temp increases $\approx 0.35°$/meter

8. Let $\boldsymbol{r}(t) = \langle -2\cos t\,,\ \sin t\rangle$

a) Find $\boldsymbol{r}\left(\frac{\pi}{6}\right)$ and $\boldsymbol{T}\left(\frac{\pi}{6}\right)$ the position and tangential vectors at the point where $t = \frac{\pi}{6}$

$$r(\pi/6) = \langle -2\cos \pi/6\,,\ \sin \pi/6 \rangle = \langle -\sqrt{3}, 1/2 \rangle$$

$$r'(t) = \langle 2\sin t, \cos t\rangle\,,\quad r'(\pi/6) = \langle 2\cdot\tfrac{1}{2}, \sqrt{3}/2\rangle = \langle 1, \sqrt{3}/2\rangle$$

$$T(\pi/6) = \frac{r'(\pi/6)}{|r'(\pi/6)|} = \frac{\langle 1, \sqrt{3}/2\rangle}{\sqrt{1+3/4}} = \frac{2}{\sqrt{7}}\langle 1, \sqrt{3}/2\rangle$$

$$= \langle 2/\sqrt{7}, \sqrt{3}/\sqrt{7}\rangle$$

not required $\longrightarrow$ $\approx \langle .756, .654\rangle$

b) Find the curvature at the point where $t = \frac{\pi}{6}$

$$r'(t) = \langle 2\sin t, \cos t\rangle \qquad r'(\pi/6) = \langle 1, \sqrt{3}/2\rangle$$

$$r''(t) = \langle 2\cos t, -\sin t\rangle \qquad r''(\pi/6) = \langle \sqrt{3}, -1/2\rangle$$

$$r'(\pi/6)\times r''(\pi/6) = \begin{vmatrix} i & j & k \\ 1 & \sqrt{3}/2 & 0 \\ \sqrt{3}, & -1/2 & 0 \end{vmatrix} = 0i - 0j + \left(-\tfrac{1}{2} - \tfrac{3}{2}\right)k = -2k$$

$$\kappa = \frac{|r'\times r''|}{|r'|} = \frac{|-2|}{(1+3/4)^{3/2}} = 2\cdot\left(\frac{2^3}{7^{3/2}}\right) = \frac{16}{7\sqrt{7}} \approx .8639$$

c) Draw a rough sketch of $\boldsymbol{r}(t) = \langle -2\cos t\,,\ \sin t\rangle$ and show the vectors $\boldsymbol{r}\left(\frac{\pi}{6}\right)$ *and* $\boldsymbol{T}\left(\frac{\pi}{6}\right)$ in your sketch.

| $t$ | $x$ | $y$ |
|---|---|---|
| 0 | $-2$ | 0 |
| $\pi/6$ | $-\sqrt{3} \approx -1.7$ | $1/2$ |
| $\pi/2$ | 0 | 1 |