Essential Problems to know for Math 172:
#1, 3, 4, 6, 8, 10, 11, 12 (short way), 13, 14, 15
If you can't do these, you will struggle in Math 172

Do not open this booklet until instructed to do so

Math 171 Final Exam

Fall 2017

Y. Petersen

Name Key Seat #

I have not given or received outside assistance on this exam.

Signature ______________________________

Points assigned:

#1:	3 pts per part,	18 pts total
#2:	4 pts per part,	8 pts total
#3:	6 pts per part,	30 pts total
#4 to 15: 12 pts each:		144 pts
Total for exam:		200 pts

1. Find the derivative of each of the following functions (18 pts total, 3 pts each):

   a) $f(x) = \sin(x^2)$ $\quad y' = \cos x^2 \cdot 2x$

   b) $f(x) = \dfrac{1}{x}$ $\quad y = x^{-1}, \; y' = -\dfrac{1}{x^2}$

   c) $f(x) = \tan^{-1} x$ $\quad y' = \dfrac{1}{1+x^2}$

   d) $f(x) = e^{5+2x}$ $\quad y' = 2e^{5+2x}$

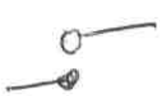
   e) $f(x) = 2x^3 - 7x - 3$ $\quad y' = 6x^2 - 7x$

   f) $f(x) = \sqrt{\ln x}$ $\quad y' = \frac{1}{2} \frac{1}{\sqrt{\ln x}} \cdot \frac{1}{x}$

2. (8 pts) a) Give an example (description or drawing) of a function which is continuous, but not differentiable.

or  ← vert. tangent

   b) Give an example (description or drawing) of a function which is integrable, but not continuous.



finite # of jump discontinuities

3. Find the limits; write "DNE" if the limit does not exist (30 pts total, 6 pts. each)

a) $\lim\limits_{x\to 3}\dfrac{x-3}{x^2-9}$ $= \lim\limits_{x\to 3}\dfrac{1}{x+3} = \dfrac{1}{6}$

b) $\lim\limits_{x\to 0^-}\dfrac{|x-5|}{x-5}$ $= \dfrac{|0-5|}{0-5} = \dfrac{5}{-5} = -1$

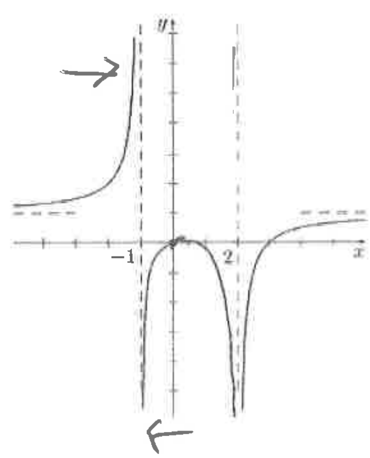
c) $\lim\limits_{x\to\infty} x^{\frac{1}{x}}$ $y = x^{1/x}$, $\ln y = \frac{1}{x}\ln x = \dfrac{\ln x}{x}$

$\lim\limits_{x\to\infty}\dfrac{\ln x}{x} = \lim\limits_{x\to\infty}\dfrac{1}{x}\cdot\dfrac{1}{1} = 0$; $y = e^0 = 1$

d) $\lim\limits_{x\to\infty}\left(x-\sqrt{x^2+x}\right)$ $\dfrac{\left(x+\sqrt{x^2+x}\right)}{x+\sqrt{x^2+x}} = \lim\limits_{x\to\infty}\dfrac{\cancel{x^2}-\cancel{x^2}-x}{x+\sqrt{x^2+x}} = \dfrac{-x/x}{\frac{x}{x}+\sqrt{\frac{x^2}{x^2}+\frac{x}{x^2}}}$

$= \dfrac{-1}{1+1} = \dfrac{-1}{2}$

e) For f(x) as shown in the graph, find:



i) $\lim\limits_{x\to -1} f(x) =$ DNE

ii) $\lim\limits_{x\to -1^+} f(x) =$ $-\infty$

4. Let $s(t) = \frac{12t}{t+1}$ be a displacement function of a moving particle. Find:

a) the instantaneous velocity at t = 1

$$s'(t) = \frac{(t+1)12 - 12t(1)}{(t+1)^2} = \frac{12}{(t+1)^2}$$

$$s'(1) = \frac{12}{2^2} = \frac{12}{4} = 3$$

$$s'(1) = 3$$

b) the average velocity of the particle from t = 0 to t = 1

$$\text{avg velocity} = \frac{\Delta s}{\Delta t} = \frac{s(1) - s(0)}{1 - 0}$$

$$= \frac{6 - 0}{1 - 0} = 6$$

5. Choose one of the following proofs:

a) Use the formal epsilon/delta definition of a limit to prove: $\lim_{x \to 3}\left(\frac{1}{3}x + 4\right) = 5$ OR

b) Prove the sum rule using the limit definition of derivative: If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

Prove for all $\varepsilon > 0$, there is a $\delta > 0$ such that!

Condition ①

If $|x-3| < \delta$ then $|\frac{1}{3}x + 4 - 5| < \varepsilon$

$|\frac{1}{3}x - 1| < \varepsilon$

$\hookrightarrow |x-3| < 3\varepsilon$

Let $\delta = 3\varepsilon$.

$|x-3| < \delta \Rightarrow |x-3| < 3\varepsilon$

and $|\frac{x}{3} - 1| < \varepsilon$

$\therefore$ There is a $\delta = 3\varepsilon$ such that Condition ① is true

$$\lim_{h \to 0} \frac{f(x+h) + g(x+h) - (f(x) + g(x))}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= d/dx\, f(x) + d/dx\, g(x)$$

product rule

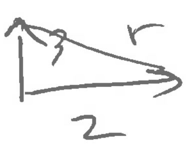
6. Find dy/dx: $x^2 + 4xy - y^2 = 10$ By implicit differentiation

$$2x + \underbrace{4x}\cdot dy/dx + y\cdot 4 - 2y \cdot dy/dx = 0$$

$$\frac{dy/dx\,(4x-2y)}{4x-2y} = \frac{-2x-4y}{} = \frac{-2(x+2y)}{4x-2y}$$

$$dy/dx = \frac{-\cancel{2}(x+2y)}{\cancel{2}(2x-y)} = \frac{-x-2y}{2x-y}$$

7. A car travels east at 60 mph on a highway and passes a road perpendicular to the highway, which leads to a farm house 3 miles north of the highway. How fast is the distance between the farm house and the car increasing when the car is 2 miles past the exit to the farm house?



$$x^2 + y^2 = r^2$$

$$2x \cdot dx/dt + 2y \cdot \overset{0}{dy/dt} = 2r\, dr/dt$$

$$dr/dt = x \cdot dx/dt \cdot \frac{1}{r} = \frac{(2)}{\sqrt{4+9}} \cdot (60\text{mph})$$

≈ 33.3 mph

8. Find the equation of the line tangent to the curve $y = 2x^2\sqrt{x}$ at the point (1, 2).

$y' = \cancel{2}x^2 \cdot \frac{1}{\cancel{2}}\frac{1}{\sqrt{x}} + \sqrt{x}\cdot 4x \;=\; 1+4=5$

$y-2=5(x-1)$

$y=5x-3$

9. For the function graphed below,

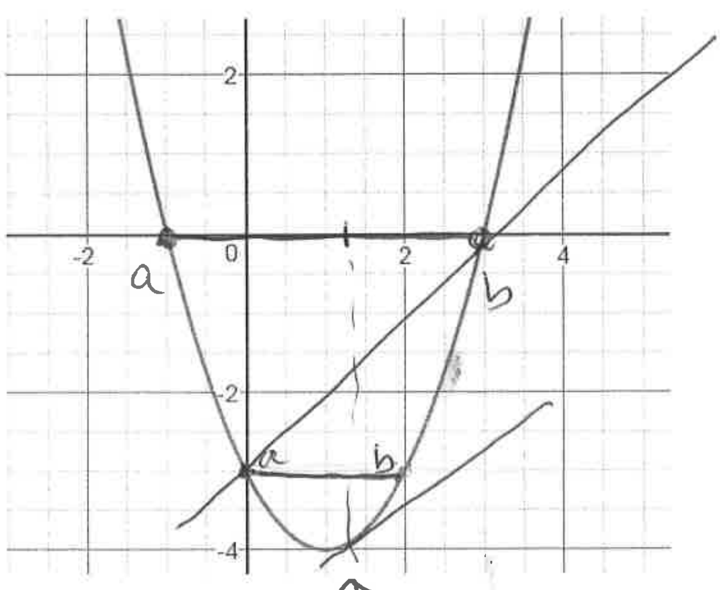
a) State a closed interval where Rolle's Theorem applies, and on the graph, show values of "a" and "b" which demonstrate the slope condition for the closed interval you have chosen.

$[-1,3]$ $\quad f(a)=f(b)=0$

Also $[0,2]$ $\Rightarrow$ any interval where y values of endpts are equal and slope between 2 pts = 0

b) Give an approximate value(s) of c that satisfy the Mean Value Theorem on [0, 3], and show the value(s) of c satisfying the slope condition of the Mean Value Theorem.

$c \approx 1.3$



$c \approx 1.3 \Rightarrow$ slope equals avg. slope

10. For $f(x) = \dfrac{4x}{x^2+4}$. Label your parts a), b), c) and d) below:

a) Give the equations of any horizontal, vertical, or slant (oblique) asymptotes.

b) Give the coordinates of any local maxima/minima.

c) Sketch the graph of f(x). Use any additional needed landmarks you need for a complete sketch, even if they're not required above.

d) (Bonus – 2 pts) Find the coordinates of all inflection points

a) no VA ; HA: $y = 0$ as $x \to \infty$

b) zero at $x = 0$

$$f'(x) = \frac{(x^2+4)(4) - 4x(2x)}{(x^2+4)^2}$$

$(2-x)(2+x)$

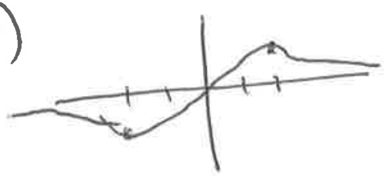
$$= 4x^2+16 - 8x^2 = \frac{4(4-x^2)}{(x^2+4)^2}$$

crit pts at $x = 2, -2$

$$f(2) = \frac{4(2)}{2^2+4} = \frac{8}{8} = 1 \qquad (2,1)$$

$$f(-2) = \frac{4(-2)}{(-2)^2+4} = \frac{-8}{8} = -1 \qquad (-2,-1)$$

c)



d)
$$f''(x) = \frac{(x^2+4)^2\cdot(-8x) - (16-4x^2)(2)(2x)(x^2+4)}{(x^2+4)^4}$$

96

$$= \frac{\cancel{(x^2+4)}(-8x^3 - 32x - 64x + 16x^3)}{(x^2+4)^{\cancel{4}\,3}} = \frac{8x(x^2-12)}{(x^2+4)^3}$$

zeros at $x = 0$, $x = \pm\sqrt{12} = \pm 2\sqrt{3}$ ① $(0,0)$ ($2\sqrt{3}$) $\pm \dfrac{8\sqrt{3}}{12+4} \to \pm\dfrac{\sqrt{3}}{2}$ $(4, \sqrt{3}/2)$ $(-4, \sqrt{3}/2)$

11. A car starts moving at time t = 0 and accelerates. Its velocity is shown in the table below.

t (seconds)	0	2	4	6	8	10	12
v (ft/second)	0	2	8	18	32	50	72

a) Estimate the distance traveled in 12 seconds using a right sum approximation.

$\Delta t = 2$

$d \approx 2(2+8+18+32+50+72) = 2(182) = 364$ ft

b) Estimate the distance traveled in 12 seconds using a left sum approximation.

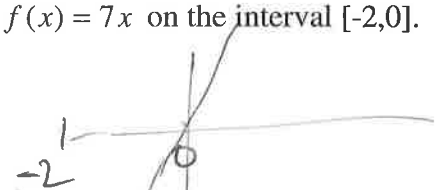
$d \approx 2(0+2+8+18+32+50) = 2(110) = 220$ ft

c) Estimate the distance traveled in 12 seconds using a midpoint approximation.

$d \approx 4(2+18+50) = 4(70) \approx 280$ ft

required long way

12. Find area under the curve using the limit (infinite) of Riemann sums for the function $f(x) = 7x$ on the interval [-2,0].



$\int_{-2}^{0} 7x\,dx$

$= \lim_{n\to\infty} \sum_{i=1}^{n} 7x = 7 \sum_{i=1}^{n} \left(-2 + \frac{2i}{n}\right) \cdot \frac{2}{n}$

The required long way →
for this test problem

$= 7\left[\frac{2}{\cancel{n}}(-2\cancel{n}) + \frac{2}{n} \cdot \frac{\cancel{2}}{\cancel{n}} \cdot \frac{\cancel{n}(n+1)}{\cancel{2}}\right]$

$= -28 + 14 = -14$

The short way!

$\int_{-2}^{0} 7x$

$= \frac{7x^2}{2}\Big|_{-2}^{0} = 0 - \frac{7(-2)^2}{2} = \frac{-28}{2} = -14$

13. a) If the volume of water in a reservoir is represented by *V(t)*, where t is the number of days since the beginning of the month, explain what $\int_{10}^{20} V'(t)dt$ represents.

Change in water volume from day 10 to day 20

b) Calculate $\int_{10}^{20} V'(t)dt$ for $V'(t) = -\frac{4}{5}t + 12$ and explain your result.

$$-\frac{4}{5}\frac{t^2}{2} + 12t \Big|_{10}^{20} = -\frac{4}{5}(200) + 240 - \left(-\frac{4}{5}\right)(50)$$

$$-120$$

$$=0$$

the volume of water at day 10 is the same as at day 20 (no difference)

14. a) Evaluate the integrals

a) $\int (2 + \sin x)dx$ $\quad 2x - \cos x\, dx + C$

b) $\int_0^{\frac{\pi}{2}} (2 + \sin x)dx$ $= 2x - \cos x \Big|_0^{\pi/2} = 2\pi/2 - 0 - (0 - 1)$

$= \pi + 1$

15. Evaluate the integrals

a) $\int \tan\theta \sec^2\theta \, d\theta$

$u = \tan \theta$

$du = \sec^2 \theta$

$\frac{\tan^2\theta}{2} + C$

b) $\int_0^1 \frac{x+1}{\left(x^2+2x+2\right)^3} dx$

$u = x^2 + 2x + 2$

$du = 2x + 2$

$= 2(x+1)$

$x = 0, \ u = 2$

$x = 1 \ u = 5$

$= \frac{1}{2} \int_2^5 \frac{du}{u^3}$

$= \frac{1}{2} \frac{u^{-2}}{-2} \Big|_2^5 = -\frac{1}{4}\left[\frac{1}{25} - \frac{1}{4}\right] = -\frac{1}{4}\left[\frac{4}{100} - \frac{25}{100}\right]$

$= -\frac{1}{4}\left(-\frac{21}{100}\right) = \frac{21}{400}$

Bonus (6 pts) Prove by induction: $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^2(n+1)^2}{4}$ (choose either fractional expression)

① $n = 1$ $\quad 1^3 = \left[\frac{1(1+1)}{2}\right]^2 = 1^2$ ✓

② $n = k$ $\quad 1^3 + 2^3 + \dots + k^3 = \left[\frac{k(k+1)}{2}\right]^2$ (assume true)

③ $n = k+1$ $\quad \left(1^3 + 2^3 + \dots + k^3\right) + (k+1)^3 = \left[\frac{(k+1)(k+2)}{2}\right]^2$?

$\frac{k^2(k+1)^2}{4} + \frac{4(k^3+3k^2+3k+1)}{4} = \frac{(k+1)\left[(k^3+k^2)\right] + 4k^3 + 12k^2 + 12k + 4}{4}$

$= \frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4} = \frac{(k^2+3k+2)(k^2+3k+2)}{4}$

$= \frac{\left[(k+1)(k+2)\right]^2}{4}$