Math 171 – Exam 3 Name

Point values range from 12 – 18 pts. per problem as shown.

1. (12 pts.) For the function $f(x) = x^3 - 3x - 1$

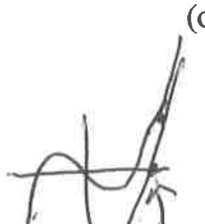
a) Use Newton's method to estimate a zero to the level of x_3 to at least 3 decimal places, starting with $x_1 = 2$

$f'(x) = 3x^2 - 3 = 3(x^2-1)$

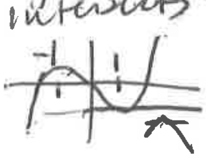
$X_2 = 2 - \frac{2^3 - 3(2) - 1}{3(2)^2 - 3} = 2 - \frac{8-6-1}{12-3} = 2 - \frac{1}{9} = 1\,8/9 = 17/9 = 1.888$

$X_3 = \frac{17}{9} - \frac{(17/9)^3 - 3(17/9)}{3(17/9)^2 - 3} \approx 1.879$

b) Give an example of a starting value of x_1 which does not converge to a root, and explain (or show through sketch) why it doesn't converge.



good tangent intersects



$X = 1$ (also $X = -1$) is a poor choice, because $f'(x) = 3(x^2-1) = 0$ when $X = 1$

When the tangent slope $= 0$, the tangent line never intersects the X-axis, never gives a zero

tangent w/ slope $= 0$ never intersects

2. (18 pts, 6 pts each part) Find the limits given, using L'Hospital's rule where applicable.

a) $\lim_{x \to 0} \frac{x - \sin x}{x^2}$ $\qquad \frac{0-0}{0}$ ok (test if L'Hospital applies)

by derivative: $\lim_{X \to 0} \frac{1 - \cos X}{2X}$ ($1 - 1/0$) $\xrightarrow{\text{derive again}} \lim_{X \to 0} \frac{-(-\sin x)}{2} = \frac{0}{2} = 0$

2. (continued) Find the limits given, using L'Hospital's rule where applicable.

b) $\lim\limits_{x \to 0} \dfrac{1+\tan^2 x}{4e^x}$

test if L'Hospital applies. No need to derive

$\dfrac{1+0}{4e^0} \rightarrow$

$= \dfrac{1+\tan^2(0)}{4e^0} = \dfrac{1}{4}$

$\dfrac{\sqrt{\infty}}{e^{\infty}} \rightarrow \infty/\infty$ - Use L'Hospitals

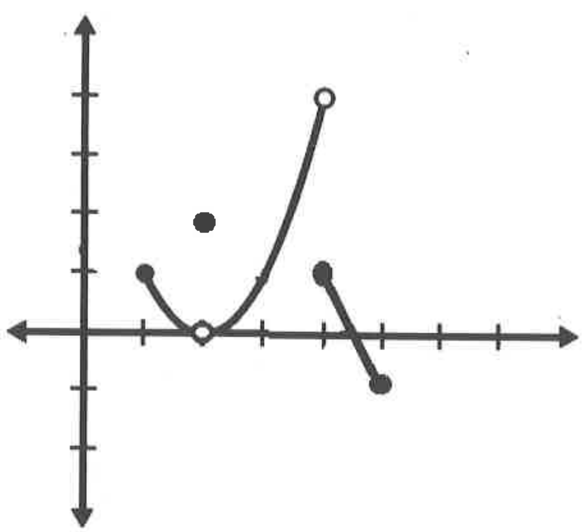
c) $\lim\limits_{x \to \infty} \sqrt{x} \cdot e^{-x}$

$= \lim\limits_{x \to \infty} \dfrac{\sqrt{x}}{e^x} \longrightarrow$ derive $f(x)$, $g(x)$

$= \lim\limits_{x \to \infty} \dfrac{\frac{1}{2}(x^{-1/2})}{e^x} = \lim\limits_{x \to \infty} \dfrac{1}{2x^{1/2}e^x}$

$\Rightarrow \dfrac{1}{\infty \cdot \infty} = 0$

3. (16 pts., 4 pts. each) Use the graph to find all local and absolute maxima and minima. Write coordinates as ordered pairs to the nearest integer (each axis mark is one unit), or "none" if no maximum or minimum exists.



A_{max} none

A_{min} (5, −1)

L_{max} (2, 2)

L_{min} none

4. (12 pts.) For the function graphed below, for the interval [-1, 2]

a) Explain why this function satisfies the conditions of the Mean Value Theorem on [-1, 2].

1) It is continuous on [-1, 2] – no gaps

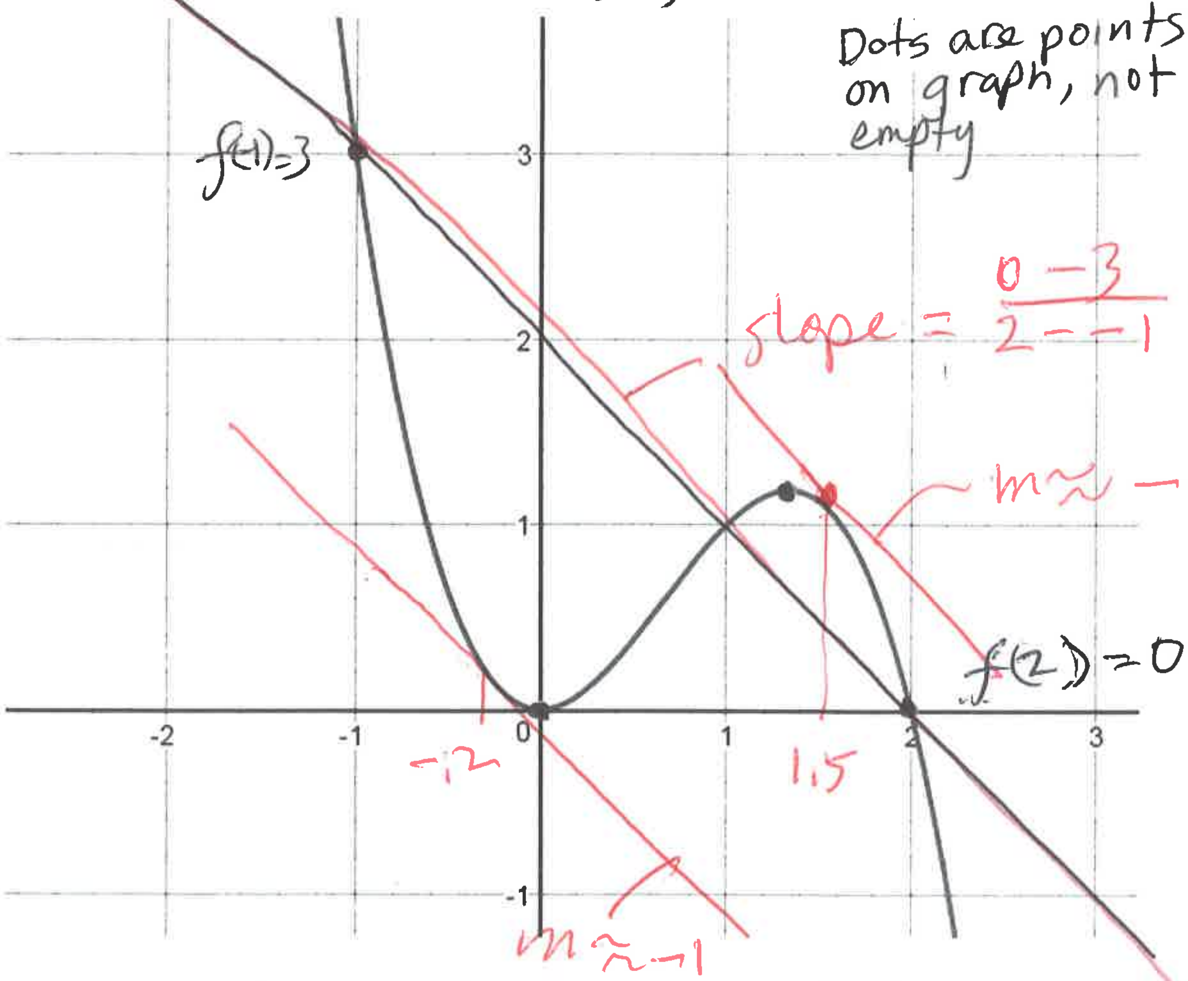
2) It is differentiable on [-1, 2] – no "corners" or vertical tangents

b) Give approximate values of "c" that satisfy the MVT (it's OK if values are "a little off" if the explanation in part c) is consistent with your numbers).

−.2, 1.5

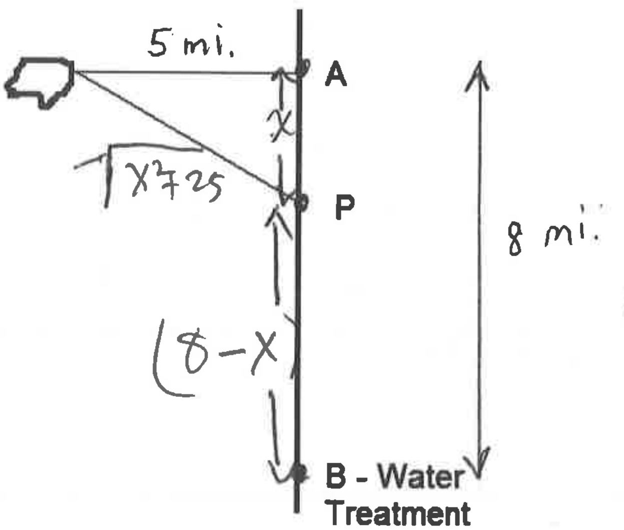
c) Explain or show on the graph how these values are related to the conclusion of the MVT.

at x = −0.2 and x = 1.5, slope ≈ −1, the slope of the secant between points (−1, 3) and (2, 0) ← the end points



b) Bonus (2 pts.) The equation of the graph above is $f(x) = -x^3 + 2x^2$. Find c values to the nearest hundredth that satisfy the MVT on [-1, 2]

5. (12 pts) Water is being supplied to an island 5 miles offshore. The nearest treated water (B) is 8 miles south of the point closest to the island (A) on a straight shoreline. It costs \$10/mile to lay pipe underwater and \$6/mile to lay pipe on land (real-life costs are in thousands of \$, but smaller numbers produce the same result). Pipe will be laid underwater to some point (P) along the shoreline. How far should P be from A to <u>minimize the cost?</u>



Find cost function
Take derivative
Find x values where
$C'(x) = 0$

(Total) Cost = cost/ft · # of feet
for both water and land

$$C(x) = 10\sqrt{x^2+25} + \overbrace{6(8-x)}^{48-6x}$$

$$f'(x) = 10 \cdot \tfrac{1}{2}\left(\frac{1}{\sqrt{x^2+25}}\right) \cdot \frac{2x}{1} - 6$$

$$= \frac{10x}{\sqrt{x^2+25}} - 6 = 0$$

$$\frac{\overset{5}{\cancel{10}}x}{\sqrt{x^2+25}} = \cancel{6}\,3 \Rightarrow \cancel{\frac{5x}{\sqrt{x^2+25}} = \frac{3}{1}}$$

sq. both sides

$$(5x) = (3\sqrt{x^2+25}) \Rightarrow 25x^2 = 9(x^2+25)$$

$$25x^2 - 9x^2 = 9 \cdot 25 = 225$$

$$16x^2 = 225$$

$$x^2 = \frac{225}{16}\ ;\ x = \sqrt{\frac{225}{16}} = \frac{15}{4} = \boxed{3\tfrac{3}{4}\text{ ft}}$$

from A to P

6. (12 pts.) For $f(x) = \dfrac{x^2}{x-1}$ ← zeros at $x=0$ ← vertical asymptotes at $x=1$

a) Find the equations of any asymptotes

$\dfrac{P(x) \text{ degree}}{Q(x) \text{ degree}} = \dfrac{2}{1} \rightarrow$ one higher → slant asymptote

$$x-1 \overline{)x^2} \quad = x+1+\cancel{\frac{1}{x-1}}\to 0$$

$$\underline{(x^2-x)}$$

$$x$$

$$\underline{-(x-1)}$$

$$+1$$

$$\boxed{\begin{array}{l}\text{slant asymptote: } y = x+1 \\ \text{VA: } x = 1 \\ \text{no HA: As } x \to \infty,\ f(x) \to \infty\end{array}}$$

b) Find the coordinates any local maxima and minima

Crit: $x=0,\ 2$

$$f'(x) = \frac{(x-1)(2x) - x^2}{(x-1)^2} = \frac{2x^2-2x-x^2}{(x-1)^2} = \frac{x^2-2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}$$

$x=0$: $f(0)=0$

$x=2$: $f(2) = 4/1 = 4$

$$\boxed{\begin{array}{l}(0,0) - \text{Lmax} \\ (2,4) - \text{Lmin}\end{array}}$$

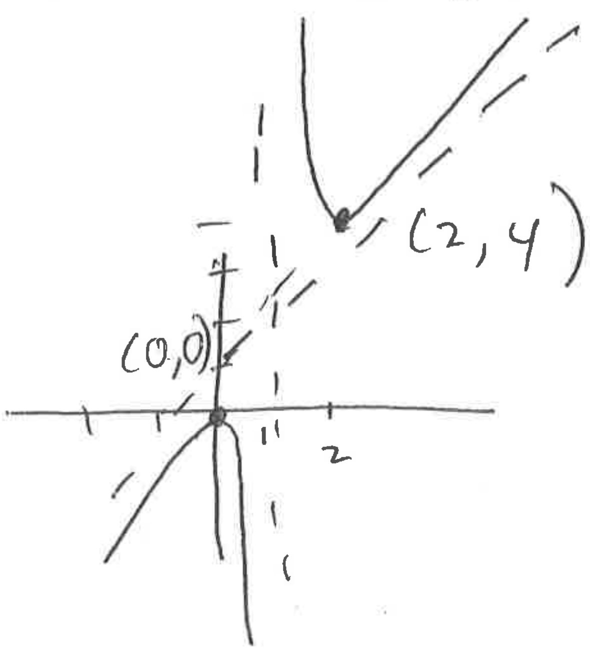
c) Find the region(s) where f(x) is concave up ~~and concave down~~

$$f''(x) = \frac{(x-1)^2 \cdot (2x-2) - (x^2-2x)(\cancel{x-1})(2)}{(x-1)^4} = \frac{\cancel{(x-1)}[2x^2-4x+2-2x^2+4x]}{(x-1)^{\cancel{4}\,3}}$$

$$= \frac{2}{(x-1)^3}$$

$$\boxed{\begin{array}{l}\text{CU: pos when } x>1 \rightarrow (1,\infty) \\ \text{CD: neg when } x<1 \rightarrow (-\infty,1)\end{array}}$$

d) Sketch the graph of f(x), using any additional needed landmarks.



Lmin: (2,4)

Lmax: (0,0)

7. (12 pts., 3 pts each part) Find the antiderivative for each derived function, $f'(x)$ below.

a) $f'(x) = 12x^2 - 4x + 7$ $\qquad f(x) = \dfrac{12x^3}{3} - \dfrac{4x^2}{2} + 7x + C$

$f(x) = 4x^3 - 2x^2 + 7x + C$

b) $f'(x) = \cos x$ $\qquad f(x) = \sin x + C$

c) $f'(x) = \dfrac{1}{1+x^2}$ $\qquad f(x) = \tan^{-1} x + C$

d)) $f'(x) = \dfrac{1}{x}$ $\qquad f(x) = \ln|x| + C$

Bonus (6 pts.) Choose ONE of the following (do "a" or "b", but not both)

a) Prove that $f(x) = x^5 + 2x + 2$ has exactly one root on its domain OR

b) Sketch the graph of $f(x) = \dfrac{7}{\sqrt{1 - \dfrac{x^2}{c^2}}}$, where c is constant, and $-c < x < c$

a) ① Use IVT to show there is a root:

$f(0) = 2$, $f(-1) = -1$. Therefore, by IVT,

There is a root, r_1, on $[-1, 0]$ where $f(r_1) = 0$

② Use Rolle's theorem to show there is no other root.

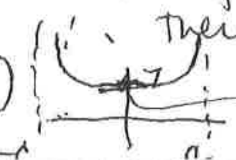
Suppose there is another root, r_2 on $(-\infty, \infty)$

Since $f(r_1) = f(r_2) = 0$, there is a "c" on $(-\infty, \infty)$

where $f'(c) = 0$.

But $f'(x) = 5x^4 + 2 \rightarrow$ always > 0

$\therefore$ There is no other root, r_2 on $(-\infty, \infty)$

b) 

Local min at $(0, 7)$, asymptotes at $x = c$, $x = -c$