Math 171 – Exam 2 Name Key

Each problem is worth 12 pts, except # 1 and #3, as noted.

1. Find the derivative of each function (3 pts each, for a total of 18 pts):

a) $f(x)=\frac{1}{x} = x^{-1}$ $\quad f'(x) = -1(x)^{-2} = -\frac{1}{x^2}$

b) $f(x) = 2^x = e^{\ln 2^x}$ $\quad f'(x) = (\ln 2)\, 2^x$

c) $y = \sec x$ $\quad f'(x) = (\sec x)(\tan x)$ (easiest from memory)

d) $g(x) = \sqrt{x} = x^{1/2}$ $\quad f'(x) = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$

e) $y = \cosh x$ $\quad f'(x) = \sinh x$

f) $f(x) = \log_{10} x = \frac{\ln x}{\ln 10}$ $\quad f'(x) = \frac{1}{\ln 10} \cdot \frac{1}{x} = \frac{1}{x \ln 10}$

2. Find the derivatives of each of the following:

a) $y = \frac{5x^3}{\sqrt{1-x^2}}$

Method 1 – Quotient Rule

$$y' = \frac{\left[(1-x^2)^{1/2} \cdot 15x^2\right] - \left[5x^3 \cdot \frac{1}{2} (1-x^2)^{-1/2} \cdot (-2x)\right]}{1-x^2}$$

$$= \underbrace{\frac{15x^2}{(1-x^2)^{1/2}} + \frac{5x^4}{(1-x^2)^{3/2}}}_{\text{OK}} = \underbrace{\frac{15x^2 - 10^4}{(1-x^2)^{3/2}}}_{\text{OK}}$$

Method 2 – log diff.

$$\ln y = \ln 5x^3 - \ln (1-x^2)^{1/2}$$

$$\ln y = 3 \ln 5x - \tfrac{1}{2} \ln(1-x^2)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{3 \cdot 5}{5x} - \frac{1}{2} \cdot \frac{1}{(1-x^2)} \cdot (-2x)\; y$$

$$\underbrace{\frac{dy}{dx} = \left[\frac{3}{x} + \frac{x}{1-x^2}\right] \cdot \frac{5x^3}{\sqrt{1-x^2}}}_{\text{OK}}$$

b) $g(x) = e^x \sin^2 3x$

Use product Rule power rule

$$g'(x) = e^x \cdot d(\sin^2 3x)/dx + de^x/dx \cdot \sin^2 3x$$

$$= e^x (2 \sin^1 3x \cdot \cos 3x \cdot 3 + \sin^2 3x)$$

$$= e^x \sin 3x\, (6 \cos 3x + \sin 3x)$$

3. Choose ONE of the following proofs or limits (10 pts)

a) Prove: If f and g are differentiable, then $\frac{d}{dx}\left[f(x)\cdot g(x)\right] = f(x)\cdot g'(x) + g(x)\cdot f'(x)$

b) Prove: $\frac{d}{dx}(\tan x) = \sec^2 x$

c) Find the limit: $\lim_{x\to 0}\frac{\cos x - 1}{\sin x}$

a) See proof in notes (handout)

b) $\tan x = \frac{\sin x}{\cos x}$ → By quotient rule!

$$d(\tan x)/dx = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

c) $\lim_{x\to 0}\frac{\cos x - 1}{\sin x} = \lim_{x\to 0}\frac{\cos x - 1}{x}\cdot\frac{x}{\sin x}$

$= \lim_{x\to 0}\frac{\cos x - 1}{x} \Big/ \lim_{x\to 0}\frac{\sin x}{x} = 0/1 = 0$

4. Find the equation of the tangent line to the curve: $x^3 + y^3 = 16$ at the point (2, 2).

By implicit differentiation, with respect to x!

$3x^2 + 3y^2 \cdot dy/dx = 0$ ← deriv. of constant

$dy/dx = -\frac{3x^2}{3y^2} = \frac{-x^2}{y^2} = \frac{-2^2}{2^2} = -1$ at (2,2)

$m = -1$

$y - 2 = -1(x - 2) = -x + 2$

$y = -x + 4$

product a

5. For $\underbrace{f(x) = x\sin x}_{}$, find the linearization, $L(x)$ when $x = \pi$ $L(x) = f(a) + f'(a)(x-a)$

$f(\pi) = \pi \cdot \sin \pi = \pi \cdot 0 = 0$

$f'(x) = x \cdot \cos x + \sin x \cdot (1) = x \cos x + \sin x$

$f'(\pi) = \pi \cos \pi + \sin \pi = \pi(-1) + 0 = -\pi$

$L(x) = 0 + -\pi(x - \pi)$

$L(x) = -\pi x + \pi^2$

Bonus (2 pts): If Δx = 0.1, find dy and Δy $\Delta x = dx = 0.1$

$dy = f'(\pi) \cdot dx = -\pi \cdot (0.1) \approx -0.3142$ ← slightly different

$\Delta y = f(\overbrace{\pi + .1}^{3.2416}) - f(\pi) \approx 3.2416(\sin 3.2416) - 0 = -0.3236$

6. Air is pumped into a spherical balloon at a rate of 9 cubic cm/sec. What is the rate of change of the radius at the instant the volume equals 36 π?

The volume of a sphere with radius r: $V = \frac{4}{3}\pi r^3$

rate of 9 cm³/sec = dV/dt

what is rate of change of radius?: $dr/dt = ?$

Instant volume = 36π : $V = 36\pi$

Derive everything (implicit) w/respect to t:

$dV/dt = \cancel{3} \cdot \frac{4}{\cancel{3}}\pi r^2 \, dr/dt$

$\cancel{9} = 4\pi \cdot \cancel{9} \cdot dr/dt$

$dr/dt = \frac{1}{4\pi}$ cm/sec

Use volume to find radius:

$V = \frac{4}{3}\pi r^3$

$36\cancel{\pi} = \frac{4}{3}\cancel{\pi} r^3$

$27 = r^3$, $r = 3$ cm

$A(t) = A_0 e^{rt}$ or $\frac{A(t)}{A_0} = e^{rt}$ $\qquad \frac{A}{A_0} = 2$

7. A sum of money invested at continuous compound interest doubles in 23 years.
   a) What is the interest rate?

Take ln of both sides

$$\frac{A(t)}{A_0} = 2 = e^{r(23)} \qquad \frac{\ln 2}{23} = \cancel{\ln e}^{23r} = \frac{23r}{23}$$

$$r = .0301 \approx 3\%$$

   b) What is the value of the account if $1000 is invested at this rate for 30 years?

$$A(t) = A_0 e^{kt}$$
$$A(t) = A_0 e^{.0301 \cdot t}$$
$$A(30) = 1000\, e^{(.0301)(30)} \approx \$2467$$

8. If $y = (\cos x)^x$, find dy/dx.

$(\cos x)^{x}$ — not a constant, so can't use power rule

Use log differentiation

Product

Not derived yet $\Rightarrow \ln y = \ln(\cos x)^x = \underbrace{x \cdot \ln(\cos x)}_{\text{Product}}$

Implicitly derive:

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}(x \cdot \ln(\cos x)) = x \cdot \frac{d}{dx} \ln(\cos x) + \ln(\cos x)$$

$$\cancel{y} \cdot \frac{1}{\cancel{y}} \cdot \frac{dy}{dx} = \left[x \cdot \frac{1}{\cos x} \cdot (-\sin x) + \ln(\cos x)\right] \cdot y$$

replace

$$dy/dx = (-x \tan x + \ln(\cos x)) \cdot (\cos x)^x$$

Bonus: (4 pts): $f(x) = \tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)$. Find $f'(x)$ and simplify to a single denominator

$$f'(x) = \frac{1}{1+x^2} + \frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot (-1 \cdot x^{-2}) = \frac{1}{1+x^2} - \frac{1}{\left(1+\frac{1}{x^2}\right)x^2}$$

$$= \frac{1}{1+x^2} - \frac{1}{x^2+1} = 0$$