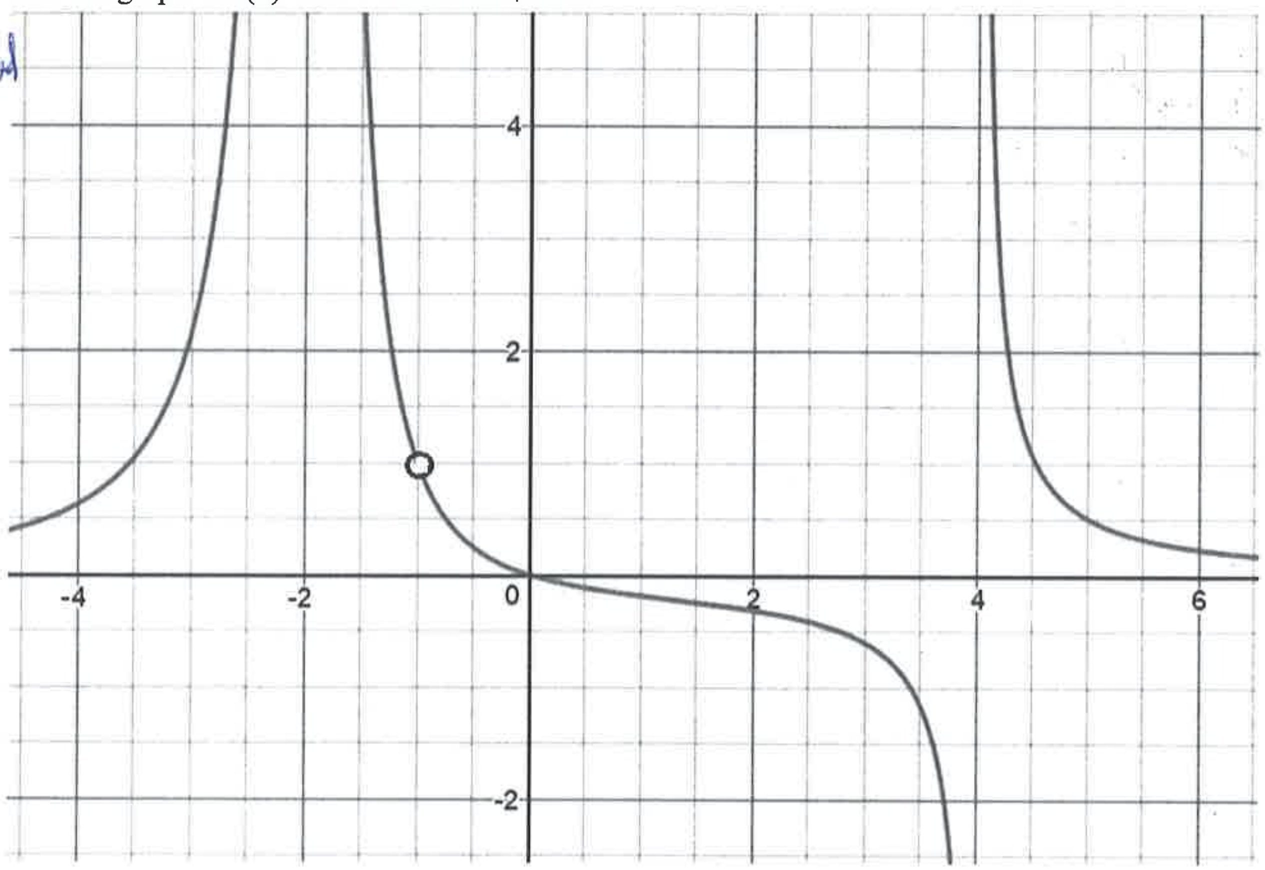
Math 171 – Exam 1 Name Key

For problems where you are asked to find the limit, if it does not exist, state "DNE."
Problems are worth 12 pts. each, except # 6 and 7, whose points are as marked.

Pages 1 and 2 are switched in the alternate version

1. Use the graph of f(x) below to find the limits



a1) $\lim_{x\to -2^+} f(x) = \infty$ a2) $\lim_{x\to -2^-} f(x) = \infty$ a3) $\lim_{x\to -2} f(x) = \infty$

b1) $\lim_{x\to -1^+} f(x) = 1$ b2) $\lim_{x\to -1^-} f(x) = 1$ b3) $\lim_{x\to -1} f(x) = 1$

c1) $\lim_{x\to 0^+} f(x) = 0$ c2) $\lim_{x\to 0^-} f(x) = 0$ c3) $\lim_{x\to 0} f(x) = 0$

d1) $\lim_{x\to 4^+} f(x) = \infty$ d2) $\lim_{x\to 4^-} f(x) = -\infty$ d3) $\lim_{x\to 4} f(x) =$ DNE

Bonus (2 pts) Write the equation of a function, f(x), whose zeros and undefined points are represented by the graph above.

$$f(x) = \frac{x(x+1)}{(x+2)^2(x+1)(x-4)}$$

2. For the function $f(x) = x^2 - 3x$

a) Find $f'(x)$

$$\lim_{h \to 0} \frac{(x+h)^2 - 3(x+h) - (x^2 - 3x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{3x} - 3h - \cancel{x^2} + \cancel{3x}}{h}$$

$$= \lim_{h \to 0} \frac{2x\cancel{h} + h^{\cancel{2}} - 3\cancel{h}}{\cancel{h}} = \lim_{h \to 0} 2x + \cancel{h}^{0} - 3 = 2x - 3$$

b) Find the equation of the line tangent to $f(x)$ at the point (1, -2)

$m = 2(1) - 3$
$= -1$

$/\!/ \; f'(1)$

$y - (-2) = -1(x - 1)$

$y + 2 = -x + 1$

$y = -x - 1$

or $f(x) = -x - 1$

3. a) Does $f(x) = e^x - \sqrt{x}$ [$3\sqrt{x}$] have a zero in the interval (1, 3)? Justify your answer.

$f(1) = e^1 - 3\sqrt{1} \approx 2.718 - 3 \approx -0.2817$

$f(3) = e^3 - 3\sqrt{3} \approx 14.89$

Yes: There is a value "c" where

$f(c) = 0$, since $-0.2817 < 0 < 14.89$

implies $1 < c < 3$

b) Name the theorem used to justify your answer above.

Intermediate value theorem

4. Use the formal (epsilon/delta) definition of limit to prove:
$\lim_{x \to 3} (2x - 5) = 1$

Find δ such that

$|x-3| < \delta \xrightarrow{\text{implies}} |2x-5-1| < \varepsilon$

$|2x-6| < \varepsilon$

$\frac{2|x-3|}{2} < \frac{\varepsilon}{2}$ choose $\delta = \varepsilon/2$

$\therefore$ If $|x-3| < \varepsilon/2$

$|2x-6| < \varepsilon$, $|2x-5-1| < \varepsilon$ is proved

5. The area of a square tile is machined to 100 cm^2, with an error tolerance of $\varepsilon = 2$ cm^2. Area, $A(x) = x^2$, where x is the tile length.
   a) Find a value of δ which produces this ε.

$\varepsilon = 2$ cm^2, so area "wiggles" from 98 to 102 cm^2

Find target value of x: $A = x^2$

$100 = x^2$, $x = 10$ cm,

Upper limit of x: $102 = x^2$, $x = \sqrt{102} = 10.0995$

$|10.0995 - 10| = .0995$

Lower limit of x: $98 = x^2$, $x = 9.8995$

$|9.8995 - 10| = 0.1005$

Choose δ = smaller error $= 0.0995$ (or less)

   b) Explain the practical meaning of δ in this context.

δ is the error in the x direction,

If x is limited to $10 - .0995 < x < 10 + .0995$

the error in area will be limited to 2 cm^2 (or less)

6. (6 pts each part)

a) Find $\lim_{x\to 0} \frac{2x^2}{x^2-9}$ $= \frac{2(0)}{0^2-9} = \frac{0}{-9} = \boxed{0}$

Direct substitution

Parts b) and c) are switched in alternate version

b) Find $\lim_{x\to\infty} \frac{2x^2}{x^2-9}$ $= \lim_{x\to\infty} \frac{\frac{2x^2}{x^2}}{\frac{x^2}{x^2} - \frac{9}{x^2}} = \lim_{x\to\infty} \frac{2}{1-\frac{9}{x^2}}$ $\frac{9}{x^2} \to 0$

Since $x\to\infty$,

use $\frac{1}{x^r}$ theorem $= \frac{2}{1} = \boxed{2}$

Method ① Calculate points

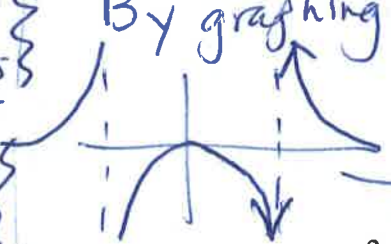
x	$f(x)$
3.01	301.5
3.001	3001.5
2.99	-289.5
2.999	-2980?

no limit

c) Find $\lim_{x\to 3} \frac{2x^2}{x^2-9}$ $= \frac{2x^2}{(x+3)(x-3)}$ zero double ← Can't remove singularity at $x=3$

asymptotes

Method ② By graphing:



$\boxed{\text{Limit DNE}}$ because different on left and right, limits don't match

7. (10 pts.) Find $\lim_{x\to -3} \frac{3-|x|}{3+x}$

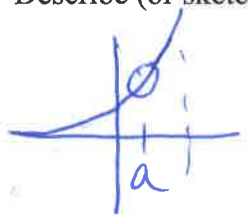
By definition

$|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$ ← This applies near $x=-3$

$\lim_{x\to -3} \frac{3-(-x)}{3+x}$

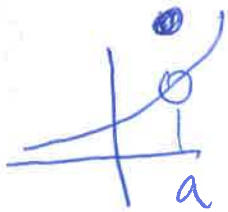
$= \lim_{x\to -3} \frac{3+x}{3+x} = \lim_{x\to -3} 1 = \boxed{1}$

8. a) Describe (or sketch) a function where $\lim_{x \to a} f(x)$ exists, but $f(a)$ does not exist.



A function with a hole at "a" has a limit, but no function value at "a"

b) Describe (or sketch) a function where $f(a)$ exists, and $\lim_{x \to a} f(x)$ exists, but $\lim_{x \to a} f(x) \neq f(a)$.



A function with a hole at "a" and a point at "a" that does not lie on the curve has a limit and a function value, but they're not equal.

c) The examples in parts a) and b) fail the criteria for what condition or property?

Continuity

Partial credit for "differentiability"

Extra credit (5 pts.)

Find $\lim_{x \to 4} \left[\frac{1}{\sqrt{x}-2} - \frac{4}{x-4} \right]$

Let $u = \sqrt{x}$; $u^2 = x$

for $x > 0$, $u > 0$

positive x

$$= \lim_{u \to 2} \left[\frac{1}{u-2} - \frac{4}{u^2-4} \right] = \lim_{u \to 2} \left[\frac{u+2}{(u+2)(u-2)} - \frac{4}{(u+2)(u-2)} \right]$$

$$= \lim_{u \to 2} \frac{\cancel{u-2}}{(u+2)\cancel{(u-2)}} = \lim_{u \to 2} \frac{1}{u+2} = \frac{1}{4}$$