The derivative of a constant is zero.

For any constant c, $\frac{d}{dx}c = 0$.

Let $f(x) = c$, for all x.

$$\frac{d}{dx}c = f'(x) = \lim_{h \to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h \to 0}\frac{c-c}{h} = \lim_{h \to 0} 0 = 0$$

The derivative of a constant times a function equals the constant times the derivative of the function.

If c is a constant and f is a differentiable function, then $\frac{d}{dx}\left[cf(x)\right] = c\frac{d}{dx}f(x)$.

Let $g(x) = cf(x)$. Then

$$g'(x) = \lim_{h \to 0}\frac{g(x+h) - g(x)}{h} = \lim_{h \to 0}\frac{cf(x+h) - cf(x)}{h} = \lim_{h \to 0} c\left[\frac{f(x+h) - f(x)}{h}\right]$$

$$= c\lim_{h \to 0}\frac{f(x+h) - f(x)}{h} = cf'(x)$$

The power rule where n is a positive integer.

If n is a positive integer, then $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$.

Let $f(x) = x^n$. Then

$$f'(a) = \lim_{x \to a}\frac{f(x) - f(a)}{x-a} = \lim_{x \to a}\frac{x^n - a^n}{x-a} = \lim_{x \to a}\left(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}\right)$$

$$= a^{n-1} + a^{n-2}a + \cdots + aa^{n-2} + a^{n-1} = na^{n-1}$$

The derivative of a sum equals the sum of the derivatives.

If f and g are both differentiable, then $\frac{d}{dx}\left[f(x)+g(x)\right]=\frac{d}{dx}f(x)+\frac{d}{dx}g(x)$.

Let $F(x)=f(x)+g(x)$. Then

$$F'(x)=\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=\lim_{h\to 0}\frac{\left[f(x+h)+g(x+h)\right]-\left[f(x)+g(x)\right]}{h}$$

$$=\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right]=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$=f'(x)+g'(x)$$

The product rule: The derivative of a product equals the first times the derivative of the second plus second times the derivative of the first.

If f and g are both differentiable, then $\frac{d}{dx}\left[f(x)g(x)\right]=f'(x)g(x)+f(x)g'(x)$.

Let $F(x)=f(x)g(x)$. Then

$$\frac{d}{dx}\left[f(x)g(x)\right]=F'(x)=\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=\lim_{h\to 0}\frac{\left[f(x+h)g(x+h)\right]-\left[f(x)g(x)\right]}{h}$$

$$=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)}{h}+\lim_{h\to 0}\frac{f(x)g(x+h)-f(x)g(x)}{h}$$

$$=\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}g(x+h)\right]+\lim_{h\to 0}\left[f(x)\frac{g(x+h)-g(x)}{h}\right]$$

$$=\left[\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right]\lim_{h\to 0}\left[g(x+h)\right]+f(x)\lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right]$$

$$f'(x)g(x)+f(x)g'(x)$$

The quotient rule: The derivative of a quotient equals the bottom times the derivative of the top minus the top times the derivative of the bottom all over the bottom squared.

If f and g are both differentiable, then $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{f'(x)g(x)-f(x)g'(x)}{\left[g(x)\right]^2}$.

Let $F(x)=\frac{f(x)}{g(x)}$. Then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=F'(x)=\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=\lim_{h\to 0}\frac{\left[\frac{f(x+h)}{g(x+h)}\right]-\left[\frac{f(x)}{g(x)}\right]}{h}$$

$$=\lim_{h\to 0}\frac{\left[\frac{f(x+h)g(x)-f(x)g(x+h)}{g(x+h)g(x)}\right]}{h}=\lim_{h\to 0}\left[\frac{f(x+h)g(x)-f(x)g(x+h)}{hg(x+h)g(x)}\right]$$

$$=\lim_{h\to 0}\left[\frac{f(x+h)g(x)-f(x)g(x)+f(x)g(x)-f(x)g(x+h)}{hg(x+h)g(x)}\right]$$

$$=\lim_{h\to 0}\frac{\frac{f(x+h)-f(x)}{h}g(x)-f(x)\frac{g(x+h)-g(x)}{h}}{g(x+h)g(x)}$$

$$=\frac{\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]g(x)-f(x)\lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right]}{\lim_{h\to 0}g(x+h)g(x)}$$

$$\frac{f'(x)g(x)-f(x)g'(x)}{\left[g(x)\right]^2}$$

The derivative of the $\sin x$ **equals the** $\cos x$**.**

$$\frac{d}{dx}\sin x=\cos x$$

Let $f(x)=\sin x$. Then

$$\frac{d}{dx}\sin x=f'(x)=\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}=\lim_{h\to 0}\frac{\sin x\cos h+\sin h\cos x-\sin x}{h}$$

$$=\lim_{h\to 0}\frac{\sin x\cos h-\sin x}{h}+\lim_{h\to 0}\frac{\sin h\cos x}{h}=\sin x\lim_{h\to 0}\frac{\cos h-1}{h}+\cos x\lim_{h\to 0}\frac{\sin h}{h}$$

$$=\sin x(0)+\cos x(1)=\cos x$$

The chain rule: Consider a composite function $f(u(x))$. It's derivative equals the derivative of f with respect to u times the derivative of u with respect to x.

If the function f has a derivative f' and the function u has a derivative $\frac{du}{dx}$, then the composite function $f(u)$ is differentiable, and $\frac{d}{dx}\left[f(u)\right] = f'(u)\frac{du}{dx}$.

Lemma

$$g'(y) = \lim_{k\to 0}\frac{g(y+k)-g(y)}{k} \Rightarrow \frac{g(y+k)-g(y)}{k} - g'(y) \to 0 \text{ as } k \to 0$$

Let $v = \frac{g(y+k)-g(y)}{k} - g'(y)$. Then $v \to 0$ as $k \to 0$.

$\Rightarrow g(y+k) = g(y) + (g'(y)+v)k$ where $v \to 0$ as $k \to 0$ This is our lemma.

$$f'(u(x)) = \lim_{h\to 0}\frac{f(u(x+h)) - f(u(x))}{h}$$
$$= \lim_{h\to 0}\frac{f[u(x)+(u'(x)+v_1)h] - f(u(x))}{h}$$
$$= \lim_{h\to 0}\frac{f(u(x)) + (f'(u(x)+v_2)(u'(x)+v_1)h - f(u(x))}{h}$$
$$= \lim_{h\to 0}\frac{(f'(u(x)+v_2)(u'(x)+v_1)h}{h}$$
$$= \lim_{h\to 0}\left[(f'(u(x)+v_2)(u'(x)+v_1)\right]$$
$$= f'(u(x))u'(x) = f'(u)\frac{du}{dx}$$

The derivative of $e^x = e^x$.

$$\frac{d}{dx}\left(e^x\right) = e^x$$

Let $f(x) = e^x$. Then

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^x(e^h-1)}{h} = e^x\lim_{h\to 0}\frac{(e^h-1)}{h} = e^x(1) = e^x$$

The derivative of $a^x = a^x \ln a$**.**

$$\frac{d}{dx}\left(a^x\right) = a^x \ln a$$

Note $a = e^{\log_e a} = e^{\ln a} \Rightarrow a^x = e^{(\ln a)x}$

Let $f(x) = a^x$. Then

$$\frac{d}{dx}\left(a^x\right) = \frac{d}{dx}\left(e^u\right) \text{ where } u = (\ln a)x$$

$$\frac{d}{dx}\left(e^u\right) = e^u \frac{d}{dx}\left[(\ln a)x\right] = e^u\left(\ln a\right) = e^{(\ln a)x} \ln a = a^x \ln a$$

The derivative of $\log_a x = \frac{1}{x \ln a}$.

$$\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x \ln a}$$

Let $y = \log_a x$. Then

$$a^y = x \Rightarrow a^y \ln a \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{a^y \ln a} = \frac{1}{x \ln a}$$